MATH 3235 Probability Theory
8/30/2022

Probability Space

$$(\Omega, \mathcal{F}, \mathbb{P})$$

Sample space — Space of events — Probability

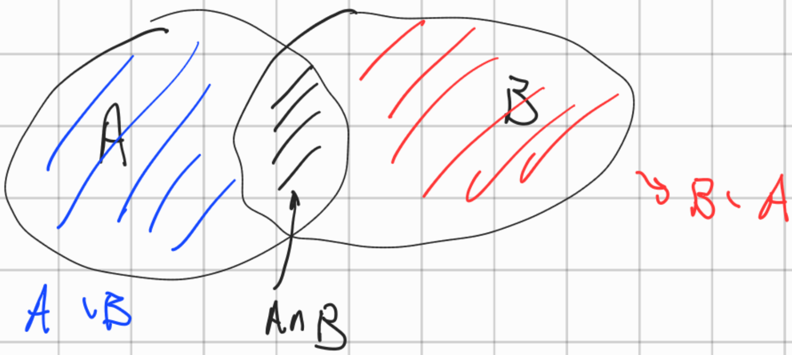
$$A \cap B = \emptyset$$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$$

What if $A \cap B \neq \emptyset$?

Set Theory:

$$A = (A \cap B) \cup (A \setminus B)$$



$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \setminus B)$$

Fact: $A \subset B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

$$A \cup B = (A \cap B) \cup (A \setminus B) \cup (B \setminus A)$$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A \cap B) + \mathbb{P}(A \setminus B) + \mathbb{P}(B \setminus A)$$

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \setminus B)$$

$$\mathbb{P}(B) = \mathbb{P}(A \cap B) + \mathbb{P}(B \setminus A)$$

$$\mathbb{P}(A) + \mathbb{P}(B) = \mathbb{P}(A \cup B) + \mathbb{P}(A \cap B)$$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) -$$

$$\mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) +$$

$$\mathbb{P}(A \cap B \cap C)$$

Conditional Probability

Medical Test $\quad p = 0.01 \quad$ FP

If you are H and Take The Test There is a prob $p = 0.01$ That the result is P. $q = 1 - p = 0.99$ result is N. If you are sick prob 1 of P and 0 of N.

Suppose That in average 1% of The population is sick.

If you get a P, are you sick?

10'000 persons and Test all of Them.

~ 100 of Them are sick and

9'900 are H

P: 100 + 99

sick FP

P : 100

prob of S given P: $\frac{100}{199} \approx 0.5$

S person is sick $\mathbb{P}(S) = 0.01$

$\mathbb{P}(P \mid S) = 1$ $\mathbb{P}(N \mid S) = 0$

$\mathbb{P}(P \mid H) = 0.01$ $\mathbb{P}(N \mid H) = 0.99$

Number of P in 10000 = 199 P

Number of S = 100 P and S

$\mathbb{P}(S \mid P) = \frac{100}{199} = \frac{100}{10000} \cdot \frac{1}{199/10000}$

$= \mathbb{P}(S \cap P) / \mathbb{P}(P)$

$\mathbb{P}(S \mid P) = \frac{\mathbb{P}(S \cap P)}{\mathbb{P}(P)}$ cond. prob.

$$\mathbb{P}(P \mid H) = \frac{\mathbb{P}(P \cap H)}{\mathbb{P}(H)}$$

$$\mathbb{P}(P) = \mathbb{P}(P \mid H)\,\mathbb{P}(H) + \mathbb{P}(P \mid S)\,\mathbb{P}(S)$$

Partition Th.

$$\mathbb{P}(S \mid P) = \frac{\mathbb{P}(P \mid S)\,\mathbb{P}(S)}{\mathbb{P}(P \mid H)\,\mathbb{P}(H) + \mathbb{P}(P \mid S)\,\mathbb{P}(S)}$$

Bayes formula.

A and B events $\quad (\mathbb{P}(B) \neq 0)$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} \frac{\mathbb{P}(A)}{\mathbb{P}(B)} =$$

$$= \mathbb{P}(B \mid A) \frac{\mathbb{P}(A)}{\mathbb{P}(B)}$$

$\mathbb{P}(A)$ $\qquad \mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

$\mathbb{P}(A \mid B) \geq 0$

$\mathbb{P}(A \mid B) \leq 1$

$$\mathbb{P}(\Omega \mid B) = \frac{\mathbb{P}(\Omega \cap B)}{\mathbb{P}(B)} = 1$$

$(\Omega \cap B \to B)$

$(A_1 \cup A_2) \cap B = (A_1 \cap B) \cup (A_2 \cap B)$

$A_1 \cap A_2 = \emptyset \qquad (A_1 \cap B) \cap (A_2 \cap B) = \emptyset$

$\mathbb{P}((A_1 \cup A_2) \cap B) = \mathbb{P}(A_1 \cap B) + \mathbb{P}(A_2 \cap B)$

$\mathbb{P}(A_1 \cup A_2 \mid B) = \mathbb{P}(A_1 \mid B) + \mathbb{P}(A_2 \mid B)$ if $A_1 \cap A_2 = \emptyset$

If $\mathbb{P}$ is a probability and $\mathbb{P}(B) \neq 0$ The

$\mathbb{P}(\cdot \mid B)$ is a probability

$\mathbb{P}(A \mid B)$ is a probability on A

B fixed ($\mathbb{P}(\cdot \mid B)$)

A is independent from B if

$$\mathbb{P}(A \mid B) = \mathbb{P}(A)$$

$$\Downarrow$$

$$\boxed{\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)}$$

$$\Downarrow$$

$$\mathbb{P}(B \mid A) = \mathbb{P}(B)$$

Def: A and B are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B).$$

$A_1 \quad A_2 \quad A_3$

$A_1 \perp\!\!\!\perp A_2$ (A_1 and A_2 indep)

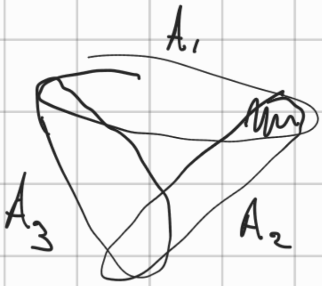
$A_2 \perp\!\!\!\perp A_3 \qquad A_1 \perp\!\!\!\perp A_3$

$\Downarrow$? In general NO!

$A_1 \perp\!\!\!\perp A_2 \cap A_3$

I need to add

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\,\mathbb{P}(A_2)\,\mathbb{P}(A_3)$$



A family of events is called independent

$A_i \qquad i \in I$

for any $J \subset I$

$$\mathbb{P}\Big(\bigcap_{j \in J} A_j\Big) = \prod_{j \in J} \mathbb{P}(A_j)$$

$A_1 \quad A_2 \quad A_3 \quad A_4$

$A_1 \quad A_2 \qquad A_3 \quad A_4$

$$A_1 \cup A_2 \underset{\text{next}}{\perp\!\!\!\perp} A_3 \cap A_4$$

H	0.49	H
	0.51	T
T	0.49	T
	0.51	H